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ABSTRACT

Mathematics educators often despair at math's austere, "abstract" reputation. This paper describes recent work in developing an application named "HyperGami," which is designed to integrate both the abstract and "real-world" aspects of mathematics by allowing children to design and construct polyhedral models and sculptures. Children use formal reasoning for computation and for creating real mathematical objects in paper. HyperGami allows students to design not only standard polyhedra, but also customized variants of those shapes. They can decorate the shapes, have the program convert three-dimensional shapes into two-dimensional shapes, and then print the decorated shape for folding into a three-dimensional paper sculpture. The paper describes the construction of a sample HyperGami sculpture, a penguin, and presents observations during a year of pilot studies with 11 volunteer elementary and middle school students. Related, current, and future work on this topic is also discussed. (Contains 12 references.) (Author/SWC)

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Paper

Polyhedral Sculpture: The Path from Computational Artifact to Real-World Mathematical Object

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Abstract

Mathematics educators often despair at the subject's austere, "abstract" reputation. This paper describes recent work in developing an application named *HyperGami*, whose purpose is to integrate both the abstract and "real-world" aspects of mathematics by

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allowing children to design and construct polyhedral models and sculptures. We describe a sample *HyperGami* sculpture, and present our observations during the past year of pilot studies with elementary and middle school students.

1. Introduction: Making Mathematical Objects "Real"

In the popular mind, mathematics has a reputation of being removed from concerns of the real world. Words such as "abstract," or worse—"cold," "remote," "austere"—are often used to describe the discipline. While indeed much of mathematical research does have an undeniably abstract nature, the subject also has its more tangible "real-world" side, as realized in the applied disciplines of engineering, architecture, and design. In this paper, we describe a computational system named *HyperGami* whose purpose is to integrate these twin "abstract" and "real-world" aspects of the mathematical enterprise. In particular, *HyperGami* provides students with an activity that weaves formal reasoning (as employed in computation) into the creation of real mathematical objects: complex three-dimensional polyhedral sculptures constructed in paper.

Before proceeding further into technical discussion of *HyperGami*, it is worth illustrating its style through an example of the type of work one can create with the software. Figure 1 shows a set of paper *HyperGami* penguins made from "custom" polyhedra (as described later in this essay). The various parts of these penguins—the heads, bodies, feet, and bowtie—are variants of particular "classical" shapes (the dodecahedron, cuboctahedron, pentagonal prism, and tetrahedron). Moreover, all the individual elements of these sculptures have been decorated using (a relatively simple subset of) the tools provided by the software.

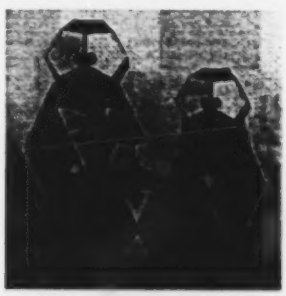

Figure 1. A family of three HyperGami polyhedral penguins.

The purpose of displaying this example is to suggest how computational media may be deployed to extend the venerable mathematical tradition of polyhedral modeling. Certainly, mathematical educators have long used paper models as tangible—and attractive—representatives of the ideas of solid geometry. (Indeed, a set of cardboard models from the 18th century may still be seen on display at the University of Göttingen [Mühlhausen 93].) But such models are often decorated—if at all—painstakingly, by hand; while kits and books for constructing polyhedral models are generally limited to the "standard," or classical, shapes. By using the *HyperGami* system, the student can design not only standard polyhedra but also customized variants of those shapes, and she can decorate those shapes in an endless variety of ways. These novel shapes may then be combined into larger composite forms such as the penguins in Figure 1. The overall effect is to extend the usual practice of polyhedral modeling into a far richer sort of activity, approaching a "mathematical art form."

The remainder of this paper is devoted to a description of the *HyperGami* software, with a particular focus on its use in the creation of polyhedral paper sculptures such as those shown in Figure 1 (we have dubbed these objects "orihedra," suggesting a blend of

origami and polyhedral modeling). The following section provides a very brief overview of the software and its major features; the third section presents a scenario showing how *HyperGami* may be used to create a sample paper sculpture. Finally, the fourth section summarizes our observations of elementary and middle school children who have used the software in the past year to create paper sculptures of their own; and we outline related work, as well as our current research directions.

2. HyperGami: A System for Creating Polyhedral Models

HyperGami is written by the authors in the MacScheme dialect of Lisp [S1], and runs on all color Macintosh computers with at least 12M of memory. The system is structured as a "programmable design environment" [Eisenberg and Fischer 94] in that it includes direct manipulation interface features supported by a full-fledged programming environment (in this case, an "application-enriched" version of the basic MacScheme language system).

A view of the *HyperGami* screen is shown in Figure 2. Briefly, the main activity of the program is the design, decoration, and construction of novel polyhedral models. The user may start with standard polyhedral shapes (such as the five regular, or "Platonic," solids), and can then alter those shapes by a variety of means, including both menu-driven commands and Scheme procedures (written in the extended Scheme dialect provided with the system). Once a new shape has been designed, the user may ask the system to "unfold" that shape automatically to produce a two-dimensional folding net pattern; this pattern may be decorated, printed out, and folded up into a new paper object. Figure 2 shows a particularly simple example: here, the user is decorating a folding net for a regular dodecahedron.

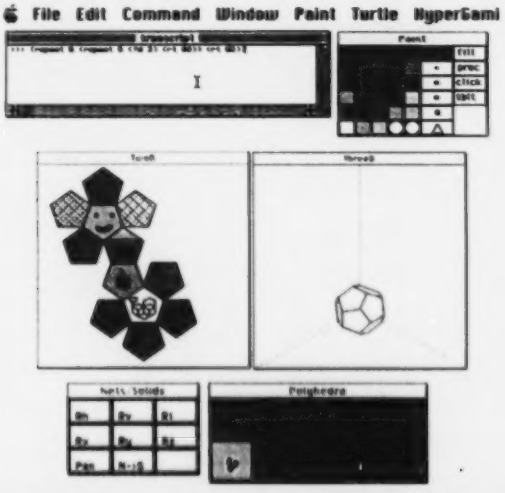

Figure 2. The HyperGami screen in the course of a sample project: the user is creating a decorated dodecahedron. The various windows are described in the text (additional optional windows are not shown here). In the figure, the user has evaluated a turtle-graphics expression, and the turtle (positioned in one of the pentagons toward the bottom) is in the course of drawing a rotated-polygon pattern.

The heart of the program is represented in Figure 2 by the two large windows labeled **TwoD** and **ThreeD**: these windows display, respectively, a two-dimensional ("unfolded") and three-dimensional ("solid") version of the shape under construction. The **Polyhedra** window provides a palette of starting shapes from which to begin one's design; the **Nets/Solids** window provides various controls for viewing the three-dimensional forms under construction; and the **Paint** window provides, among other things, a selection of standard graphics tools for (e.g.) changing the color or pen width in the process of decorating a folding net. Finally, the **transcript** window is the interface to the application's Scheme interpreter.

Figure 2 depicts a scenario in which a user is creating a decorated model of a dodecahedron. Here, the user has begun the design process by selecting the dodecahedron icon from the **Polyhedra** palette; the system displays a folding net for the solid and a three-dimensional view of the solid in the **TwoD** and **ThreeD** windows, respectively. The user now decorates the folding net using a range of techniques (as shown in Figure 2): regions may be filled with solid colors or patterns, decorated via "customized fill procedures," and painted "by hand" (using the mouse), among other possibilities. In the figure, the user is shown in the course of employing still another technique—turtle graphics—for decorating one of the regions of the dodecahedron's folding net: here, one of the pentagons in the folding net is decorated with a rotated-polygon figure [Abelson and diSessa 80].

This description of *HyperGami* has been unavoidably brief; important features for "linking" the two- and three-dimensional views of solids, as well as many linguistic features of the system's Scheme interpreter, have had to be ignored. (For more detail, see [Eisenberg and Nishioka 96].) In the following section, we will in any event go beyond the current example, in which a standard dodecahedron was constructed, and show how new customized forms may be created.

3. Designing a Polyhedral Sculpture with HyperGami

Creating original paper sculptures with *HyperGami* typically entails the design and construction of novel polyhedral forms derived from simpler shapes. In this section, we illustrate the types of *HyperGami* operations that can be used to create new solids in the context of a sample project—namely, constructing the penguins shown in Figure 1. Note that this section will focus only on the mathematical techniques needed to create new ingredients for sculptures—and not the design skills needed to conceive of those sculptures. We will touch on the latter issue toward the end of this essay.

3.1 The Penguin's Body: "Stretching" a Solid

In creating the penguin's body, we begin with the cuboctahedron—a semiregular solid (i.e., one whose faces are composed of more than one type of regular polygon—in this case, six squares and eight equilateral triangles). The cuboctahedron can be selected through *HyperGami*'s **Polyhedra** palette; it is also accessible through a larger optional palette of the so-called "Archimedean" solids.

Figure 3 shows the idea of how we would like to proceed in making the body of the penguin: at the left, we have the standard cuboctahedron, while at center we have a new variant of that solid, created by "stretching" the original shape along its vertical axis.

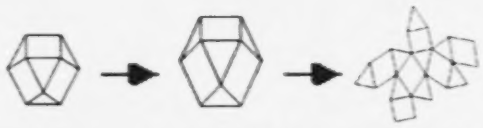

Figure 3. "Stretching" a cuboctahedron into a taller shape, which is then unfolded.

To perform this operation in *HyperGami*, we first select the cuboctahedron; then we select a menu choice that allows us to apply several types of linear maps (rotations and scales) to our already-chosen solid. In this particular instance we stretch the cuboctahedron by 1.4 along the z-axis. Once this choice is made, the system now creates a "stretched" version of the cuboctahedron, and computes a two-dimensional folding net for the solid, as shown in Figure 3. It should be noted that the "unfolding" problem—the task of calculating a folding net from a three-dimensional polyhedron—is quite complex.[Croft *et al.* 91] *HyperGami*'s unfolding algorithm is based on a strategy of heuristic search, and does not always succeed in producing a folding net for arbitrary solids; but it has yet to fail on a convex solid and has succeeded in generating patterns for a large number of nonconvex shapes as well (see [Eisenberg and Nishioka 96] for further discussion).

Once the new folding net for the "stretched" cuboctahedron has been generated as in Figure 3, it may be decorated in *HyperGami* to produce a suitable penguin body. (For the Figure 1 sculptures, two of the triangles were decorated in white—to suggest the penguin's chest and belly—and the rest were decorated in black.)

3.2 The Penguin's Head: "Capping" the Face of a Solid

The head of the penguin in Figure 1 is a variant of the dodecahedron, created by "capping" one face of that solid. The idea is suggested by Figure 4: we take the original dodecahedron and make a new solid by adding a single vertex *v* outside one of the faces of that dodecahedron, drawing new edges to *v*. The result is a solid that looks rather like a head with a beak.

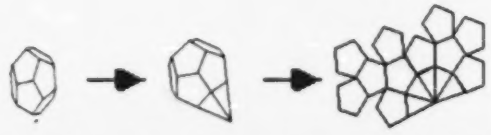

Figure 4. A dodecahedron (left) is "capped" to form a new shape which is then "unfolded" (at right).

In this case, the technique for generating a capped solid would be performed using *HyperGami*'s Scheme language operations: a function named `add-cap-vertex` is used to generate a new solid from an older one. Here is the expression used to generate the shape at the center of Figure 4:

```
(add-cap-vertex *current-solid-object* (read-a-current-
  solid-face) 0.7)
```

The `add-cap-vertex` function takes three arguments. The first argument is a solid (in this case, our "current solid"—the dodecahedron—which has the default name `*current-solid-object*` indicating that it was the last solid object selected); the second argument is a specific face of that solid (here, the `read-a-current-solid-face` procedure allows us to select that face with the mouse); and a "cap height"

indicating how far the new vertex should be from the center of the capped face. When this expression is evaluated in *HyperGami*'s language interpreter, the system prompts us to select a face of the dodecahedron, and returns the solid shown at center in Figure 4. This new form may now be unfolded; and the folding net may be decorated to suggest a penguin's head. (In the case of the Figure 1 sculptures, the beak was done in orange, the face pentagons in white—with hand-drawn eyes—and the rest of the head in black.)

3.3 The Penguin's Feet: Creating Custom Prisms

The feet of the penguins in Figure 1 are designed by creating customized prisms. *HyperGami* allows the user to generate a prism based upon an arbitrary polygon—whether that polygon is hand-drawn, created by specifying a sequence of points through a dialog box, or created by language operations. The pentagon used for the penguin's feet was in fact created by a Scheme expression:

```
(define penguin-foot-polygon
(make-closed-polygon
(make-point -0.5 0)(make-point 0.5 0) (make-point 1.1 1.86)
(make-point 0 2.5) (make-point -1.1 1.86)))
```

This polygon was "turned into" a prism by the following expression:

```
(make-general-prism-solid-object penguin-foot-polygon 0.5)
```

The `make-general-prism-solid-object` function takes two arguments: a base polygon, and a prism height. The resulting solid was unfolded and (for Figure 1) decorated in orange.

3.4 The Penguin's Bow Tie

The bow tie of the large penguin in Figure 1 is simply composed of two tetrahedra; the tetrahedron is in fact one of the five regular solids and may be selected (along with its folding net) directly from *HyperGami*'s **Polyhedra** palette. The tie of the Figure 1 sculpture was decorated in red.

Overall, then, the penguin sculptures in Figure 1 represent several important techniques for generating new solids in *HyperGami*: applying linear maps (such as "stretching" maps) to solids, "capping" faces, and creating solids (such as prisms) from hand-drawn or language-specified polygons. *HyperGami* includes still other techniques for generating solids not employed in this example: techniques for "vertex truncation" (essentially, "slicing away" a portion of a solid near a vertex [Holden 71]), for joining together two solid at a face, for creating solids from data structures (lists of vertices, edges, and faces), and for applying certain types of nonlinear maps to solids. The result is a rich "algebra of polyhedra" through which an endless range of new forms may be explored.

4. Experiences with Students; Related Work; Current, and Future Research Directions

4.1 Experiences with Students

During the past year, we initiated our first pilot tests of the *HyperGami* software with a total of eleven volunteer elementary and middle school children ranging in age from 8 to 13. The students worked with us as individuals or in pairs, coming to weekly or biweekly sessions that lasted one to three hours. Not all these children's projects

involved creating paper sculptures with *HyperGami*; some projects focused on origami figures (the software may also be used for this purpose [Eisenberg and Nishioka 94]). All of the children did choose, at some point, to create polyhedral figures (though not all of these were full-fledged "sculptures" representing nonmathematical objects like the penguins in Figure 1).

Those several children who did create their own "orihedra" employed sophisticated elements in doing so. One 11-year-old girl used antiprisms and a truncated icosahedron to create a paper hippopotamus; a 10-year-old boy experimented with an elongated icosahedron to create an "alien head"; and a 13-year-old boy expressed his fondness for dinosaurs by building a polyhedral sculpture of a brontosaurus that made use of antiprisms and a "stretched dodecahedron."

In some cases, children have begun with our basic solids and used the software to create their own custom decorations. While working with our "pre-stored" polyhedra, the children can concentrate on learning to use the decorative features of the system without the added complexity of manipulating the solid. This serves as a gentle introduction to the software in which the child arrives at a satisfying finished product. For example, during her earliest interactions with the system, an 8-year-old currently working with us has made penguins (of which she is considerably proud) by filling the outlines of our penguin nets with her own designs.

In designing their own custom polyhedra, students have the opportunity to rethink, and personalize, solid geometry. One student who had worked with the software began by describing "basic shapes ... like cubes" and then continued, "I think everything builds off of these shapes. They are sort of like the primary colors, you know, you mix to get different colors from the three primary colors and you get different shapes with the solid shapes ...." Another student, pursuing a biological analogy, stated that "there are many, totally many kinds of polyhedra out there, but they all start from a simple shape, which is basically their 'genes' ...."

4.2 Related, Current, and Future Work

HyperGami reflects the influence of a number of other efforts, both in mathematics education and in educational computing. Its subject matter is shared with a variety of materials designed for polyhedral modeling, including excellent books by Holden [71] and Wenninger [71] and kits such as the marvelous *Zometool* [BioCrystal]; but, as noted earlier, these materials include no computational component, and they focus on polyhedral modeling for the "classical" shapes (rather than customized shapes) and for strictly "mathematical" purposes. More directly, the elementary-school software package *Shape Up!*[S2] also includes folding nets (though only for a very small selection of preselected solids); and Suzuki [87], as far as we know, was the first practitioner of using polyhedral elements to create larger structures (though his elements were largely undecorated). On the philosophical side, *HyperGami* shares with such software efforts as LEGO/Logo [Resnick 93] both an interest in integrating computational and "real-world" artifacts, as well as an interest in embedding programming languages within applications.

These themes of integration, as mentioned in the previous sentence, are currently motivating our ideas about further research work. At present, we are creating a "children's mathematics book"—an alphabet book of polyhedral sculptures showing the type of work that can be pursued with the system. This book is also intended to explore an issue raised earlier in this essay—namely, the question of how composite sculptures may be conceived and designed (as opposed to the more "nuts-and-bolts" question, addressed in section 3, of how individual parts may be constructed). The alphabet book may thus be interpreted as a catalog of design techniques—what shapes may be used to represent "heads," what shapes act as good "limbs," and so forth.

In the near term, we are working to devise assessment measures to determine what sorts of cognitive progress (or problems) children experience in working with *HyperGami*, particularly in the area of mathematical visualization; many of the ideas of this research are derived from the earlier important work of Piaget and Inhelder [48]. Broadly speaking—and for the longer term—we would like to pursue an agenda of combining computational media with a range of handicrafts (including, perhaps, ceramics, waxwork, puppetry, and so forth); our hope in doing so is that computation, and mathematics, can be viewed as a little less "austere" and a bit more as disciplines that inform both the work of hand and mind.

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Software

[S1] *MacScheme*. Lightship Software. Available through Academic Distributing, Dewey AZ.

[S2] *Shape Up!* Sunburst Communications, Inc. Pleasantville, NY.

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